Math 142 Lecture 20 Notes

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1 Free Products and the Seifert-van Kampen Theorem

1.1 Free products and free groups

So far, we have proven the following "almost-classification."

Theorem 1.1. *If S is a closed surface, then*

$$S \cong S^2, \qquad S \cong T^2 \# \cdots \# T^2, \qquad \textit{or} \qquad S \cong \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2.$$

We want to prove that these are all distinct. Let's give these names.

Definition 1.1. For $g \in \mathbb{N}$, let

$$S_g := \underbrace{T^2 \# \cdots \# T^2}_{g}, \qquad N_g := \underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{g}.$$

We call g the *genus* of the surface.

We will prove that genus is well-defined by showing that S^2, S_g, and N_g are all different. The idea is to calculate $\pi_1(S)$ and show that these are different for these surfaces. We know that:

$$\pi_1(S^2) \cong 1, \qquad \pi_1(T^2) = \mathbb{Z}^2$$
$$\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}^2, \qquad \pi_1(K) = \pi_1(N_2) \cong \langle r, u \mid rur = u \rangle .$$

First, let's review some group theory. We can generate a group by a presentation, which includes generators and relations between them.

Example 1.1. Here is a group with two generators and one relation.

$$\left\langle a_1, a_2 \mid a_1 a_2 a_1^{-1} a_2^{-1} = 1 \right\rangle \cong \mathbb{Z}^2.$$

Definition 1.2. Let $G = \langle a_1, \ldots, a_n \mid r_1 = 1, \ldots, r_m = 1 \rangle$ and $G' = \langle b_1, \ldots b_{n'} \mid s_1 = 1, \ldots, s_{m'} = 1 \rangle$ be finitely generated groups. Then the *free product* of G and G' is

$$G * G' = \langle a_1, \ldots, a_n, b_1, \ldots b_{n'} \mid r_1 = 1, \ldots, r_m = 1, s_1 = 1, \ldots, s_{m'} = 1 \rangle .$$

Definition 1.3. The *free group on n generators* is the group $F_n = \langle a_1, \ldots, a_n \rangle$ (no relations).

The free group on 1 generator is $F_1 \cong \mathbb{Z}$. By induction, we see that the free group on n generators is $F_n \cong F_{n-1} * \mathbb{Z} \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n}$.

1.2 The Seifert-van Kampen theorem

Recall a theorem we proved earlier.

Theorem 1.2. *If $X = A \cup B$ with A and B open, simply connected, and path-connected and $A \cap B$ path-connected, then $\pi_1(X) \cong 1$.*

This is a special case of a more general result.

Theorem 1.3 (Seifert-van Kampen[1])**.** *Let $X = A \cup B$ with A and B open and path-connected, $p \in A \cap B$, $A \cap B$ be path-connected, and let*

$$i_A : A \cap B \to A, \qquad i_B : A \cap B \to B$$

be the inclusion maps. Then

$$\pi_1(X, p) \cong \frac{\pi_1(A, p) * \pi_1(B, p)}{N},$$

where N is the smallest normal subgroup containing the elements $(i_A)_*(g)[(i_B)_*(g)]^{-1}$ for all $g \in \pi_1(A \cap B, p)$.

The reason we want to quotient out by this subgroup is that we want to say that $(i_A)_*(g)[(i_B)_*(g)]^{-1}$ is trivial in $\pi_1(X, p)$. That is, $(i_A)_*(g) = (i_B)_*(g)$. We have to manually insert this relation because the free product of G and G' does not include any relations relating elements of G to elements of G'.

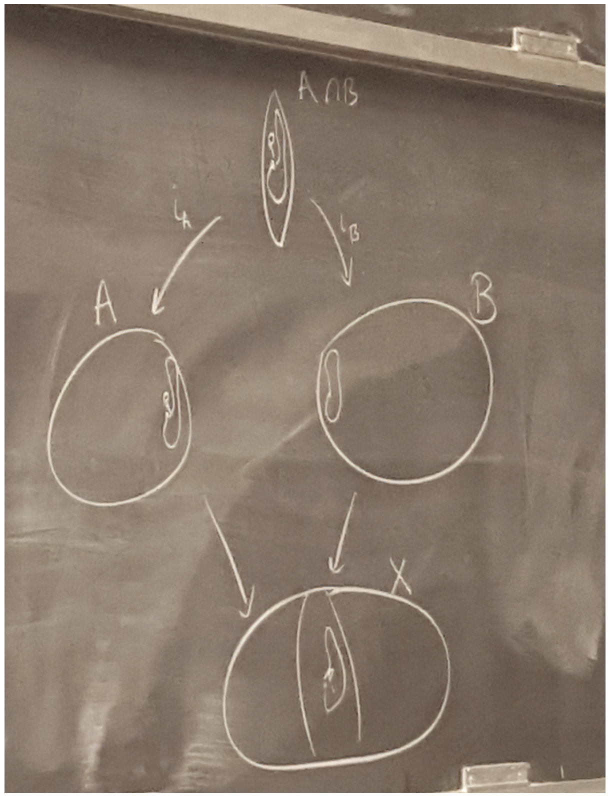
So if

$$\begin{aligned} \pi_1(A, p) &= \langle a_1, \ldots, a_n \mid r_1 = 1, \ldots, r_m = 1 \rangle, \\ \pi_1(B, p) &= \langle b_1, \ldots b_{n'} \mid s_1 = 1, \ldots, s_{m'} = 1 \rangle \\ \pi_1(C \cap B, p) &= \langle g_1, \ldots g_\ell \mid t_1 = 1, \ldots, t_k = 1 \rangle, \end{aligned}$$

then

$$\begin{aligned} \pi_1(X, p) = \langle a_1, \ldots, a_n, b_1, \ldots b_{n'} \mid & r_1 = 1, \ldots, r_m = 1, s_1 = 1, \ldots, s_{m'} = 1, \\ & (i_A)_*(g_1) = (i_B)_*(g_1), \ldots, (i_A)_*(g_\ell) = (i_B)_*(g_\ell) \rangle. \end{aligned}$$

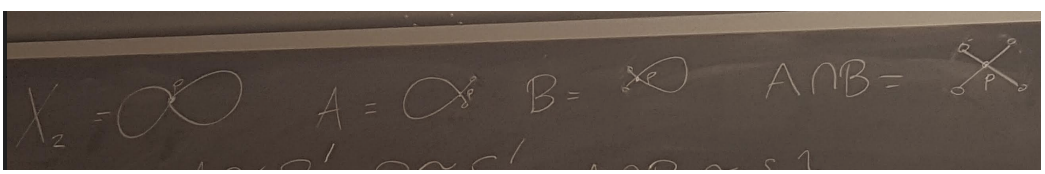
[1]This was apparently proven independently by both Seifert and van-Kampen. Sometimes, it is just called the van Kampen theorem.



1.3 Applications of the Seifert-van Kampen theorem

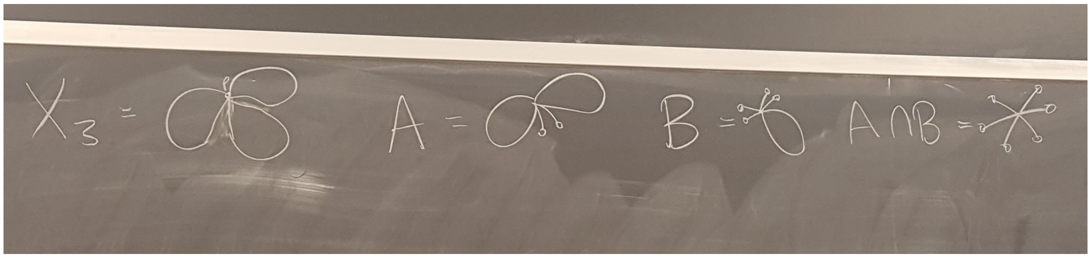
We will not prove the Seifert-van Kampen theorem, but here are some examples.

Example 1.2. Let X_2 be the 1 point union of two circles, and split into A and B as follows.



Then $A \simeq S^1$, $B \simeq S^1$, and $A \cap B \simeq \{p\}$. Since $\pi_1(A \cap B) \cong 1$, the normal subgroup $N = 1$. So $\pi_1(X_2, p) \cong \pi_1(A, p) * \pi_1(B, p) \cong \mathbb{Z} * \mathbb{Z} \cong F_2 = \{a_1, a_2\}$. The element $a_i = [\sigma_i]$, where σ_i is a path from p to p that goes once around the i-th circle.

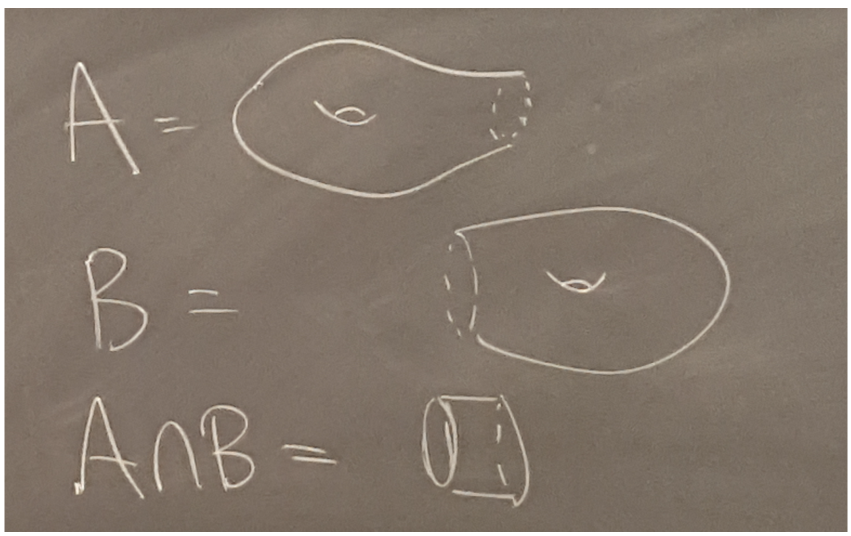
Let X_3 be the 1 point union of three circles, and split into A and B as follows.



We know that $A \simeq S_2$, $B \simeq S^1$, and $A \cap B \simeq \{p\}$. As before, $\pi_1(A \cap B, p) \cong 1$, so $N = 1$. So $\pi_1(X_3, p) \cong \pi_1(X_2) * \pi_1(S^1) \cong F_2 * \mathbb{Z} \cong F_3$.

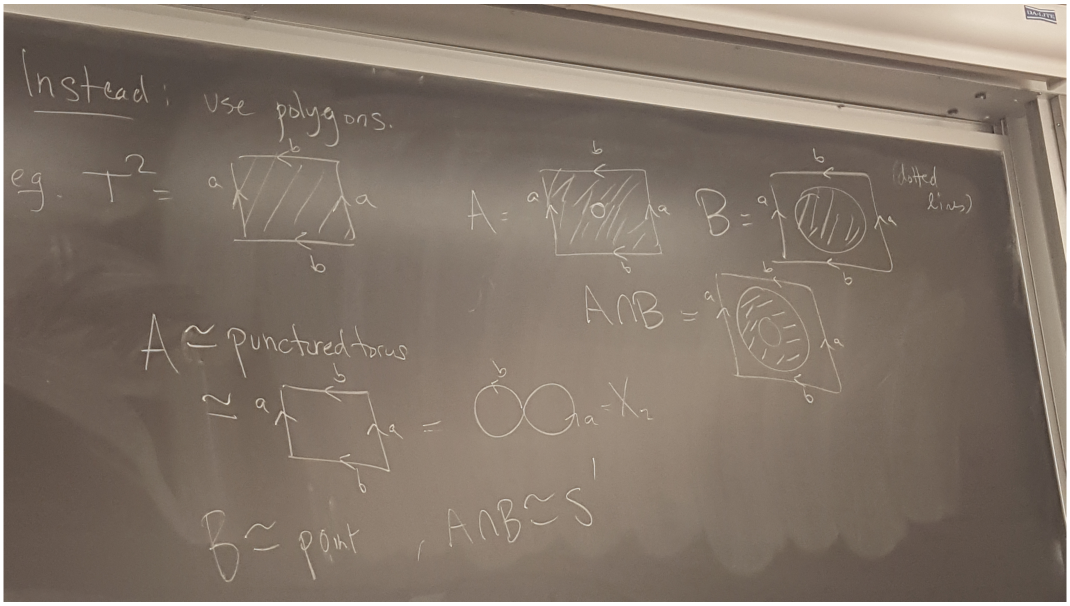
Similarly, by induction, if X_n is the 1 point union of n circles, then $\pi_1(X_n, p) \cong F_n = \langle a_1, \dots, a_n \rangle$, and $a_i = [\sigma_i]$, where σ_i is a path p to p that goes around the i-th circle once.

Example 1.3. We can form $X = S_2$ from two punctured tori.



This can be a bit confusing, so for surfaces, we will instead use polygons.

Example 1.4. Let's decompose the torus into a punctured torus and a disc.



As we did on a homework, A deformation retracts to the edges (by widening the hole), which is actually the one-point union of two circles. B deformation retracts to a single point, and $A \cap B \simeq S^1$. We have that

$$(i_A)_* : \underbrace{\pi_1(A \cap B)}_{\cong \mathbb{Z}} \to \underbrace{\pi_1(B)}_{\cong 1} \qquad \text{sends } n \mapsto 1,$$

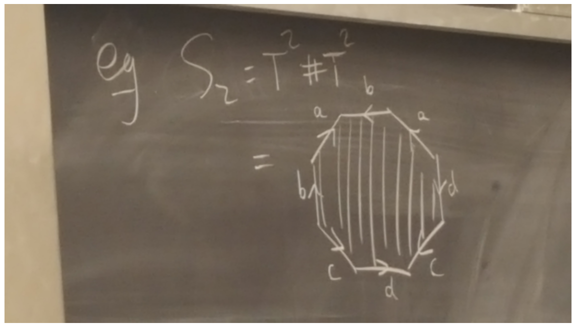
$$(i_B)_* : \underbrace{\pi_1(A \cap B)}_{\cong \mathbb{Z}} \to \underbrace{\pi_1(A)}_{\cong \langle a,b \rangle} \qquad \text{sends } 1 \mapsto aba^{-1}b^{-1},$$

which goes counterclockwise around the square. So

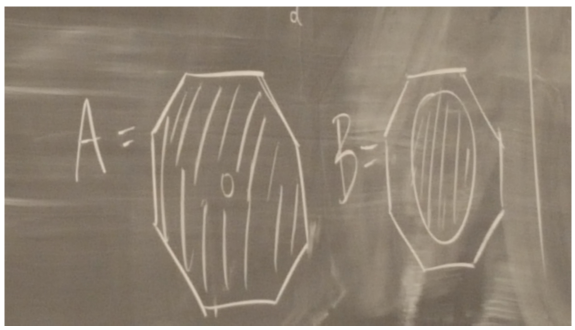
$$\begin{aligned}
\pi_1(T^2) &\cong \frac{\pi_1(A) * \pi_1(B)}{N} \\
&\cong \langle a, b \mid (i_A)_*(1) = (i_B)_*(1) \rangle \\
&= \left\langle a, b \mid aba^{-1}b^{-1} = 1 \right\rangle \\
&= \langle a, b \mid ab = ba \rangle \\
&\cong \mathbb{Z}^2.
\end{aligned}$$

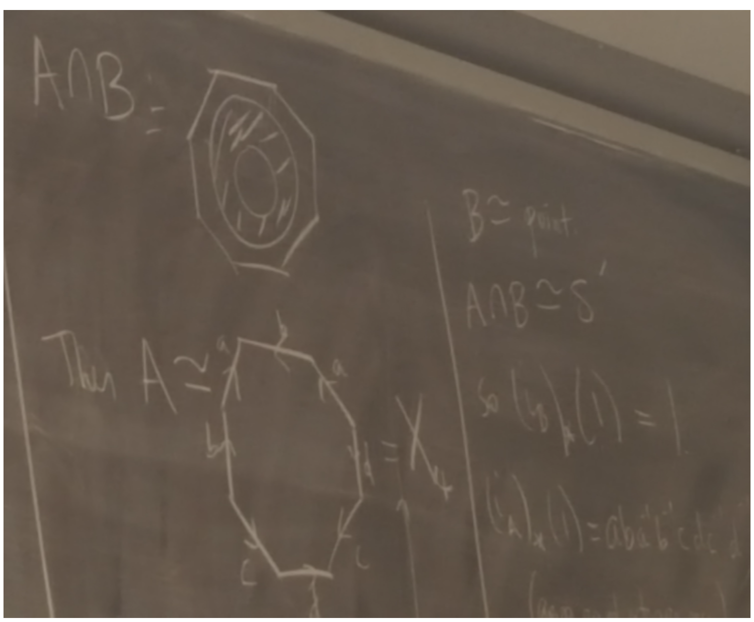
This is the third way we have calculated $\pi_1(T^2)$. The first was that we treated T^2 as $S^1 \times S^1$, and the second was that we treated T^2 as the orbit space $\mathbb{R}^2/\mathbb{Z}^2$.

Example 1.5. Look at $S_2 = T^2 \# T^2$. The single-cell cellular decomposition for S_2 is



Define A and B similarly to how we did for the torus.



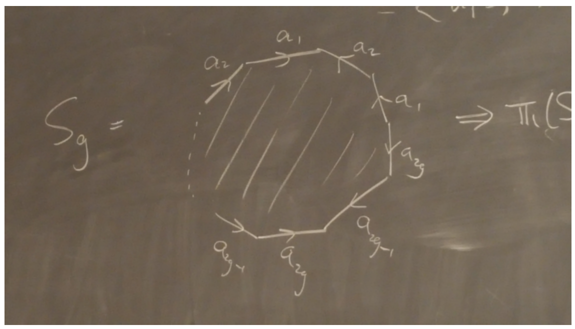


Then A deformation retracts onto the edge, which is X_4, the one-point union of 4 circles. B deformation retracts to a point, and $A \cap B \simeq S^1$. So $(i_B)_*(1) = 1$, and $(i_A)_*(1) = aba^{-1}b^{-1}cdc^{-1}d^{-1}$ (going around the octagon once). So

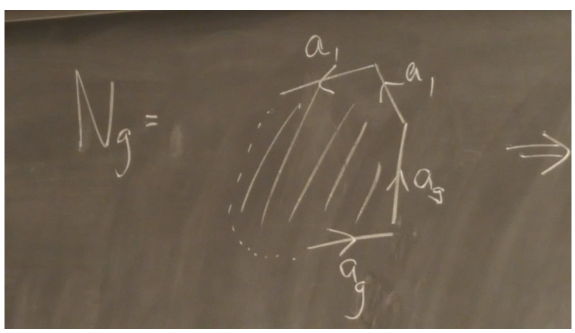
$$\begin{aligned}
\pi_1(S_2) &\cong \frac{\langle a, b, c, d\rangle * 1}{N} \\
&\cong \langle a, b, c, d \mid (i_A)_*(1) = (i_B)_*(1)\rangle \\
&\cong \left\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \right\rangle,
\end{aligned}$$

which is not a group we recognize. In general, we can get

$$\pi_1(S_g) \cong \langle a_1, \ldots, a_{2g} \mid a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1} = 1\rangle.$$



Example 1.6. We can do the same thing with N_g.



We get that

$$\pi_1(N_g) \cong \left\langle a_1, \ldots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 = 1 \right\rangle.$$

How do we know if any of these groups are the same? We will abelianize them.